Lecture 6 20-09-18

§ Homogeneous eqt with constant coefficient

$$ay'' + by' + cy = 0$$

with $a, b, c \in \mathbb{R}$, and $a \neq 0$.

Q: How to find fundamental set of solution y_1, y_2 with $W(y_1, y_2)(t) \neq 0$?

Trial: Let's try $y = e^{rt}$ and we have $(ar^2 + br + c)e^{rt} = 0$.

To solve the ODE, we need

$$ar^2 + br + c = 0$$

Def: We let the characteristic equation to be

$$\boxed{ar^2 + br + c = 0.} \quad \text{---} (*)$$

Solution: $r = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

We will separate the study of the eqt into:

Case i): ($b^2-4ac>0$) two distinct real root r_1, r_2

Case ii): ($b^2-4ac<0$) two complex root $r, \bar{r}$

Case iii): ($b^2-4ac=0$) repeated root r.

§ distinct real roots: r_1, r_2

We let $y_1 = e^{r_1 t}$, $y_2 = e^{r_2 t}$

Check: $W(y_1, y_2)(t) = \det\begin{pmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{pmatrix}$

$$= (r_2 - r_1) e^{(r_1+r_2)t} \neq 0$$

$\Rightarrow$ General solution:

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Example: $y'' + 9y' + 20y = 0$

$\Rightarrow$ char. eqt. $r^2 + 9r + 20 = 0$.

$\Rightarrow \quad (r+4)(r+5) = 0$

$r_1 = -4, \quad r_2 = -5$

$\Rightarrow$ General sol: $y(t) = C_1 e^{-4t} + C_2 e^{-5t}$

§ Distinct complex root:

- If $b^2 - 4ac < 0$, let $r = \lambda + i\mu$

  where $\lambda = \frac{-b}{2a}$, $\mu = \frac{\sqrt{4ac - b^2}}{2a}$

Idea: • e^{rt}, $e^{\bar{r}t}$ is a "fundamental set" of solution.

• However, they are complex-valued

Def: • For a complex-valued function

$y(t) = u(t) + i\,v(t)$ with u, v real-valued

• We define $\boxed{y'(t) = u'(t) + i\,v'(t)}$

Prop: Consider $y'' + p(t)y' + q(t)y = 0$

with $p(t), q(t)$ real-valued

$y(t)$ satisfy eqt $\Leftrightarrow$ both $u(t), v(t)$ satisfy the eqts

Pf: $y'' + p(t) y' + q(t) y$

$$= (u'' + p(t)u' + q(t)u) + i(v'' + p(t)v' + q(t)v)$$

$\therefore$ L.H.S $= 0 \Longleftrightarrow \begin{cases} u'' + p(t)u' + q(t)u = 0 \\ v'' + p(t)v' + q(t)v = 0 \end{cases}$

Observation:

- If we let $z = e^{rt} = e^{t\lambda}(\cos\mu t + i\sin\mu t)$

  (we use the formula $e^{i\theta} = \cos\theta + i\sin\theta$)

  $$\bar{z} = e^{\bar{r}t} = e^{t\lambda}(\cos\mu t - i\sin\mu t)$$

- $z, \bar{z}$ solution to $ay'' + by' + cy = 0$

  $\because \ (e^{rt})' = \lambda e^{\lambda t}(\cos\mu t + i\sin\mu t)$

  $$+ e^{\lambda t}(-\mu\sin\mu t + i\mu\cos\mu t)$$

  $$= (\lambda + i\mu)\, e^{\lambda t}(\cos\mu t + i\sin\mu t)$$

  $$= r e^{rt}.$$

Similarily: $(e^{\bar{r}t}) = \bar{r} e^{\bar{r}t}.$

- For real-valued solution:

By the Lemma, we have
$$u(t) = e^{\lambda t}\cos\mu t$$
$$v(t) = e^{\lambda t}\sin\mu t$$

Both are solutions to the equation

- $$u'(t) = e^{\lambda t}(\lambda\cos\mu t - \mu\sin\mu t)$$
$$v'(t) = e^{\lambda t}(\lambda\sin\mu t + \mu\cos\mu t)$$

$W(u,v)(t) =$

$$\det\begin{pmatrix} e^{\lambda t}\cos\mu t & e^{\lambda t}\sin\mu t \\ e^{\lambda t}(\lambda\cos\mu t - \mu\sin\mu t) & e^{\lambda t}(\lambda\sin\mu t + \mu\cos\mu t) \end{pmatrix}$$

$$= e^{2\lambda t}\left[\cos\mu t(\lambda\sin\mu t + \mu\cos\mu t) - \sin\mu t(\lambda\cos\mu t - \mu\sin\mu t)\right]$$

$$= \mu e^{2\lambda t} \neq 0.$$

$\Rightarrow$ u, v is a fundamental set of solution to the equation.

<u>General solution</u>: $e^{\lambda t}(c_1\cos\mu t + c_2\sin\mu t)$

Example:

$$y'' + 2y' + 10y = 0$$

$$b^2 - 4ac = 4 - 40 = -36$$

$$\rightarrow \quad \lambda = -1, \quad \mu = 3.$$

General solution: $y(t) = e^{-t}(c_1 \cos 3t + c_2 \sin 3t)$

§ Repeated root:

- Suppose $b^2 - 4ac = 0$, then we have $r = \frac{-b}{2a}$

  By earlier discussion, $\Rightarrow y_1(t) = e^{rt}$ is a solution

- We let eqt: $y'' + \underbrace{\frac{b}{a}}_{=p} y' + \underbrace{\frac{c}{a}}_{=q} y = 0.$

  Let $W(t) := \exp\left(-\int p\,dt\right) = e^{-pt}.$

- By the discussion in previous lecture we solve

$$y_1 z' - y_1' z = W(t)$$

i.e.

$$e^{rt} z' - r e^{rt} z = e^{-pt}.$$

$$z' - rz = e^{-(p+r)t}$$

$$\left(p + r = \frac{b}{a} - \frac{b}{2a} = \frac{b}{2a} \right)$$

$$z' - rz = e^{rt}$$

$$\Rightarrow \quad (e^{-rt} z)' = 1$$

$$\Rightarrow \quad e^{-rt} z = t + \cancel{c}$$

Chosen to be zero

$$z = te^{rt} = te^{\frac{-b}{2a}t}$$

$\therefore$ General sol:

$$y(t) = C_1 e^{\frac{-b}{2a}t} + C_2 t e^{\frac{-b}{2a}t}$$

Example:

$$y'' + 4y' + 4y = 0$$

Char. eqt: $r^2 + 4r + 4 = 0$

$$\Rightarrow (r+2)^2 = 0 \quad \text{repeated root}$$

$\Rightarrow$ General sol:

$$y(t) = C_1 e^{-2t} + C_2 t e^{-2t}$$